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A Short Proof of Coherence Theorem for Monoidal Categories

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Abstract : The concept of a monoidal category is fundamental and widely applied in fields such as computer science and physics, including programming language theory, natural language processing and quantum physics. The Coherence Theorem is one of the key theorems that captures the essential properties of monoidal categories. While a proof of this theorem was originally given by Saunders Mac Lane, it is somewhat difficult to follow and omits many details. In 2010, Trujillo published a paper that provided a detailed proof of the Coherence Theorem. In this paper, we present a more concise proof of the Coherence Theorem using a method different from that of Trujillo.

Key words :coherence theorem, monoidal category, binary word, general associative law, free monoid

1. Introduction

Monoidal category is a fundamental concept widely used in pure mathematics, computer science, and physics. The Coherence Theorem is a key result that captures the most basic property of monoidal categories. Although a proof of this theorem is provided in the book [1] by Saunders Mac Lane, the proof is somewhat difficult to follow and omits many details. Later, in 2010, Trujillo published a paper [2] that presented a detailed proof of the Coherence Theorem.

In this paper, we provide a concise and short proof of the Coherence Theorem using a different approach from that of Trujillo. In particular, we define all arrows in a category **W** of binary words to be either basic arrows or compositions of basic arrows, and prove that for any $v, w \in ob(\mathbf{W})$ of the same length, there exists a unique arrow in **W** from v to w. Using this theorem and the definition of morphisms in a monoidal category, we derive a proof of the Coherence Theorem. Finally, as an application of the Coherence Theorem, we demonstrate the general associative law for monoids. The Coherence Theorem implies theorem of construction of free monoid in monoidal category.

2. Preliminaries

2.1 Monoidal Category

Definition 1.

A monoidal category $\langle \mathbf{C}, \square, e, \alpha, \lambda, \varrho \rangle$ *is defined as follows.*

1. **C** *is a category.*
2. $\square$ *is a bifunctor on* **C***,* $\square : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$.
3. $e \in ob(\mathbf{C})$
4. α *is a natural isomorphism,* $\alpha : \square \circ (1 \times \square) \to \square \circ (\square \times 1)$.
5. λ *is a natural isomorphism,* $\lambda : \square \circ (e \times 1) \to 1$.
6. ϱ *is a natural isomorphism,* $\varrho : \square \circ (1 \times e) \to 1$.
7. $\lambda_e = \varrho_e : e \square e \to e$.
8. *For any* $a, b, c, d \in ob(\mathbf{C})$*, the diagram in Fig. 1 commutes.*
9. *For any* $a, b \in ob(\mathbf{C})$*, the diagram in Fig. 2 commutes.*

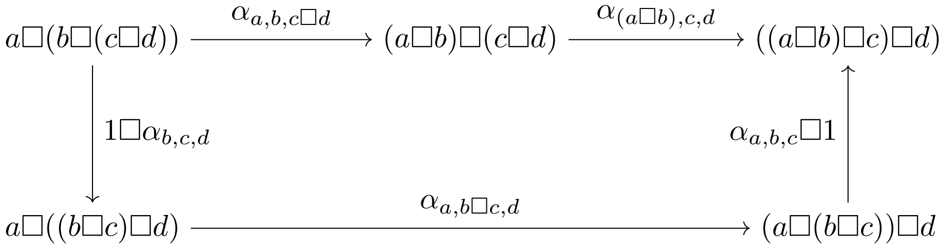

Fig.1　Commutativity about α in monoidal category

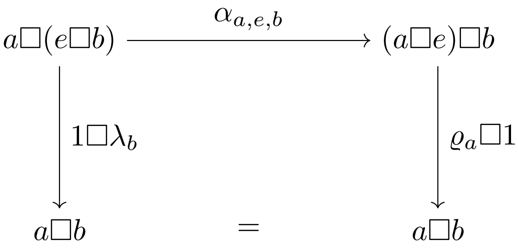

Fig.2　Commutativity about λ, ϱ in monoidal category

Here, 1 denotes the identity functor on $\mathbf{C}$. $a\Box b$ denotes $\Box(a,b)$ and $a\Box(e\Box b)$ denotes $\Box\circ(1\times\Box)(a,e,b)$. Therefore, $a\Box(b\Box(c\Box d))$ represents $\Box\circ(1\times\Box)\circ(1\times 1\times\Box)(a,b,c,d) = \Box\circ(1\times(\Box\circ(1\times\Box)))(a,b,c,d)$.

Definition 2.
Let $\langle \mathbf{C}, \Box, e, \alpha, \lambda, \varrho\rangle$ and $\langle \mathbf{B}', \Box', e', \alpha', \lambda', \varrho'\rangle$ be monoidal categories. A morphism of monoidal categories
$T : \langle \mathbf{C}, \Box, e, \alpha, \lambda, \varrho\rangle \to \langle \mathbf{B}', \Box', e', \alpha', \lambda', \varrho'\rangle$ is defined as follows.

1. *$T : \mathbf{C} \to \mathbf{B}'$ is a functor.*
2. *For any $a, b \in ob(\mathbf{C})$, any $f, g \in Hom(\mathbf{C})$,*
   $T(a\Box b) = T(a)\Box' T(b)$,
   $T(f\Box g) = T(f)\Box' T(g)$.
3. *$T(e) = e'$.*
4. *For any $a, b, c \in ob(\mathbf{C})$,*
   $T(\alpha_{a,b,c}) = \alpha'_{T(a),T(b),T(c)}$.
5. *For any $a \in ob(\mathbf{C})$,*
   $T(\lambda_a) = \lambda'_{T(a)}, T(\varrho_a) = \varrho'_{T(a)}$.

With these morphisms, **Moncat** can be formed. The objects are small monoidal categories and the arrows are morphisms of monoidal categories.

2.2 Category of Binary Words

Definition 3.
A binary word of length $n \in \mathbb{N}$ is defined by general inductive definition as follows.

1. *e_0 is a binary word of length 0.*
2. *$(-)$ is a binary word of length 1.*
3. *v, w are binary words of lengths m and n, respectively, then the $v\Box w$ is a binary word of length $m+n$.*

For example, $(-)\Box(e\Box(-)), ((-)\Box e)\Box(-)$ are binary words of length 2.

Definition 4.
A monoidal category $\langle \mathbf{W}, \Box, e_0, \alpha, \lambda, \varrho\rangle$ of binary words is defined as follows.

1. *ob$(\mathbf{W})$ is a set of binary words.*
2. *$\Box$ and e_0 are the same as those defined in Definition 3.*
3. *All arrows in $\mathbf{W}$ are either basic arrows or compositions of basic arrows.*

Definition 5.
The definition of basic arrow in $\langle \mathbf{W}, \Box, e_0, \alpha, \lambda, \varrho\rangle$ is as follows.

1. *The components of $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \varrho, \varrho^{-1}$ and 1 are basic arrows.*
2. *If β is a basic arrow, then $\beta\Box 1, 1\Box\beta$ are basic arrows.*

In the above Definition. 5, 1 denotes idetity. We present several example of basic arrow. $1\Box\alpha_{u,v,w}, (1_u\Box\alpha_{u,v,w})\Box 1_v, 1_u\Box(\lambda_v\Box 1_w), (\varrho_u\Box 1_v)$ are basic arrows. According to the definition of a functor, the identity is mapped to the identity, and hence $1_u\Box(1_v\Box 1_w) = 1_u\Box 1_{(v\Box w)} = 1_{u\Box(v\Box w)}$ holds. Therefore, $(1_u\Box\alpha_{u',v',w'})\Box(1_v\Box 1_w) = (1_u\Box\alpha_{u',v',w'})\Box 1_{v\Box w}$ is also basic arrow. Note that in the $\mathbf{W}$, the only arrow whose domain and codomain are equal is the identity by Definition 4.

Definition 6.

The rank ρ of a binary word w is defined as follows.

1. $\rho(e_0) = 0$
2. $\rho(-) = 0$
3. *For any $v, w \in ob(\mathbf{W})$, $\rho(v\Box w) = \rho(v) + \rho(w) + length(w) - 1$*

For example, $\rho((-)\Box((-)\Box(-))) = 1$, $\rho(((-)\Box(-))\Box(-)) = 0$. $\rho(w) = 0$ means that all pairs of parentheses in w start at the front.

3. Coherence Theorem

To prove Coherence theorem, we prove the following main theorem.

Theorem 1.

For any two binary word $v, w \in ob(\mathbf{W})$ of the same length, there is only one arrow $v \to w$.

It was previously stated that when $v = w$, the morphism from v to w is the identity. Therefore, in the following, we consider the case where $v \neq w$. To prove this theorem, we first prove several lemmas. Here, let $w^{(n)} \in \mathbf{W}$ be the unique binary word of length n which has all pairs of parentheses starting in front and does not involve e_0.

Lemma 1.

For any element $v, w \in ob(\mathbf{W})$ of the same length which do not involve e_0. Then there exists only one arrow from v to w.

Lemma 2.

For any element $w \in ob(\mathbf{W})$ which does not involve e_0, let n be the length of w, there exists a unique composition of directed arrows from w to $w^{(n)}$ except when $w = w^{(n)}$.

Proof of Lemma 2. Let $w \in ob(\mathbf{W})$ be length n that does not involve e_0. A directed arrow is defined as a basic arrow that contains components of α, λ, ϱ and does not contain components of $\alpha^{-1}, \lambda^{-1}, \varrho^{-1}$. Here, since we have assumed that w does not contain e_0, any directed arrow from w to $w^{(n)}$ consists only of components of α and 1, and does not contain components of λ or ϱ. There is a canonical directed path which means composition of directed arrows from w to $w^{(n)}$ successively moving outermost parentheses to the front by components of α. For example, from $((-)\Box((-)\Box(-)))\Box((-)\Box(-))$ to $w^5 = ((((-)\Box(-))\Box(-))\Box(-))\Box(-)$, Fig. 3 shows the canonical directed path.

We prove by induction on the $\rho(w)$ that every arrow from w to $w^{(n)}$ is equal to the composition of arrows contained in canonical directed path except in the case where $w = w^{(n)}$. If $\rho(w) = 0$, then $w = w^{(n)}$. When $\rho(w) = 1$ and $w \neq w^{(n)}$, it follows that $w = w^{(n-2)}\Box((-)\Box(-))$, and a directed arrow from w to $w^{(n)}$ is only component of α. For $\rho(w) \geq 2$, let $w = u\Box v$, and as shown in Fig. 4, let f and g be arbitrary directed arrows with domain w. Also in Fig. 4, let $can_i (i = 1, 2, 3)$ denote the canonical directed path. Since f and g are directed arrows, it follows that $\rho(u') \leq \rho(w)$ and $\rho(v') \leq \rho(w)$.

In this case, if there exist an object x, directed arrows h and j such that $h \circ f = j \circ g$, then by the inductive hypothesis $can_2 \circ h = can_1$ and $can_2 \circ$

$$((-)\Box((-)\Box(-)))\Box((-)\Box(-))$$

$$\Big\downarrow\ \alpha_{((-)\Box((-)\Box(-))),(-),(-)}$$

$$(((-)\Box((-)\Box(-)))\Box(-))\Box(-)$$

$$\Big\downarrow\ (\alpha_{(-),(-),(-)}\Box 1)\Box 1$$

$$((((-)\Box(-))\Box(-))\Box(-))\Box(-)$$

Fig.3 An example of canonical directed path

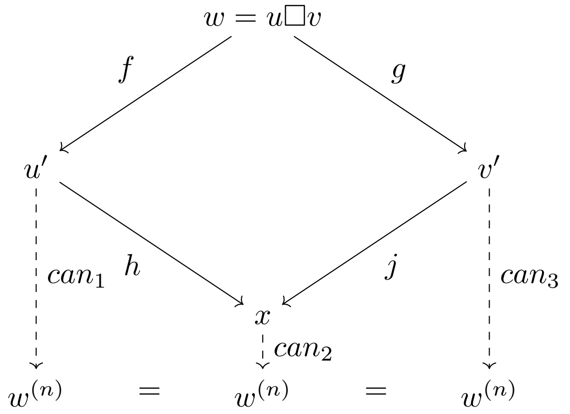

Fig.4　Composition of directed arrows

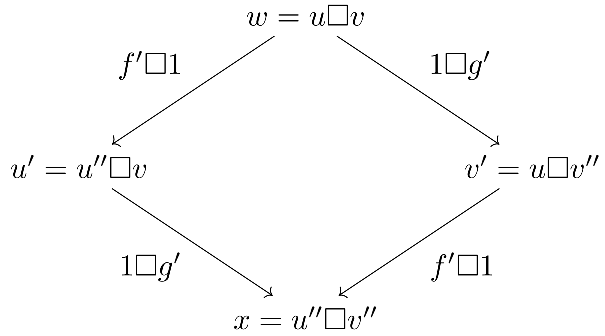

Fig.5　Directed arrows in the Case 1

$j = can_3$ holds. It implies $can_1 \circ f = can_3 \circ g$, and thus there exists a unique directed arrow from w to $w^{(n)}$. For each possible combination of f and g, we prove that $h \circ f = j \circ g$ or $can_1 \circ f = can_3 \circ g$. If $f = g$, then $can_1 \circ f = can_3 \circ g$. When $f \neq g$, f and g are one of the following. Here, both f' and g' are assumed to be directed arrows.

Case 1. $f = f' \Box 1$ and $g = 1 \Box g'$
Case 2. $f = 1 \Box f'$ and $g = g' \Box 1$
Case 3. $f = f' \Box 1$ and $g = g' \Box 1$
Case 4. $f = 1 \Box f'$ and $g = 1 \Box g'$
Case 5. Exactly one of f or g is an component of α.

In Case 1 above, as shown in the Fig. 5, $(1 \Box g') \circ (f' \Box 1) = f' \Box g' = (f' \Box 1) \circ (1 \Box g')$. Similarly, in Case 2 above, $(g' \Box 1) \circ (1 \Box f') = f' \Box g' = (1 \Box f') \circ (g' \Box 1)$.

In Case 3 above, we prove this by induction on the length n of w. If $n \leq 3$, $f = g$. If $n = 4$, Fig. 6 is commutative by the Condition. 8 of Definition. 1. If $n \geq 5$, since $length(u) \geq length(u')$ and $length(u) \geq length(u')$ in Fig. 7, by the inductive hypothesis, there are arrows h', j' which commutes Fig. 7. Therefore, the assertion also holds in the Case 3 above.

In Case 4 above, the proof is similar to that of the Case 3. The unique directed arrow from w to v is obtained by switching u and v in $w = u \Box v$ in Fig. 7, and accordingly adjusting the position of f', g', h', j' in each arrow of 7.

In Case 5 above, When f in Fig. 4 is equal to α, Fig. 8 and Fig. 9 commute by the naturality of α, and Fig.10 commutes by the Condition 8 of Definition 1, which asserts that (W) is a monoidal category. When g in Fig.4 is equal to α, the horizontally mirrored versions of the paths in Fig. 8, 9 and 10 result in equal compositions of directed arrows

This completes the proof of Lemma 2. □

Proof of Lemma 1. Let $w \in ob(\mathbf{W})$ be length n that does not involve e_0. A inverse directed arrow is defined as a basic arrow that contains components of $\alpha^{-1}, \lambda^{-1}, \varrho^{-1}$ and does not contain components of α, λ, ϱ. Then there exists a unique composition of inverse directed arrows from $w^{(n)}$ to w except when $w = w^{(n)}$. As shown in the proof of Lemma 2, there exists a unique composition β of directed arrows from w to $w^{(n)}$. Since β^{-1} is composition of inverse directed arrow, there exists at least one composition of inverse directed arrows from $w^{(n)}$ to w. Assume that there exist two compositions of inverse directed arrows β^{-1} and γ^{-1} from $w^{(n)}$ to w. Then, β and γ are compositions of direcred arrows from w to $w^{(n)}$, and by Lemma 2, we have $\beta = \gamma$. Therefore, $\beta^{-1} = \gamma^{-1}$. For any element $v, w \in ob(\mathbf{W})$ of the same length n which do not involve e_0. Then there exists at least one arrow from v to w. It is given by the composition of directed arrows from v to $w^{(n)}$ followed by the composition

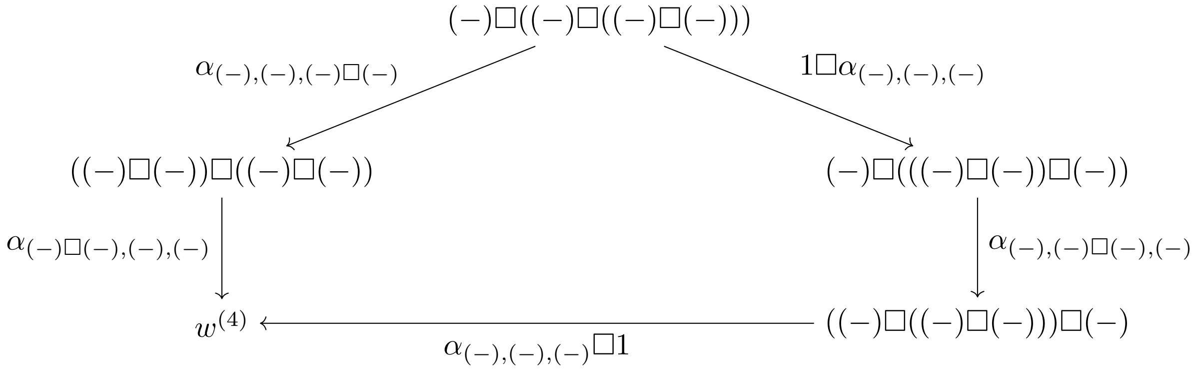

Fig.6 Commutativity of α in $\mathbf{W}$

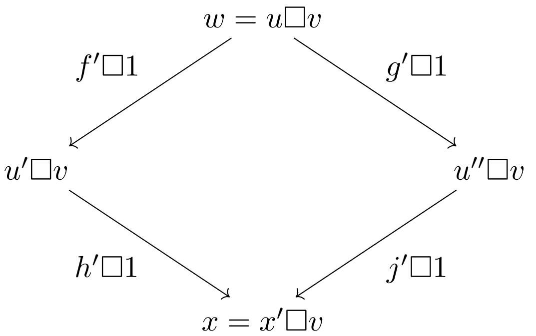

Fig.7 Directed arrows in the Case 3 and length of $w \geq 6$

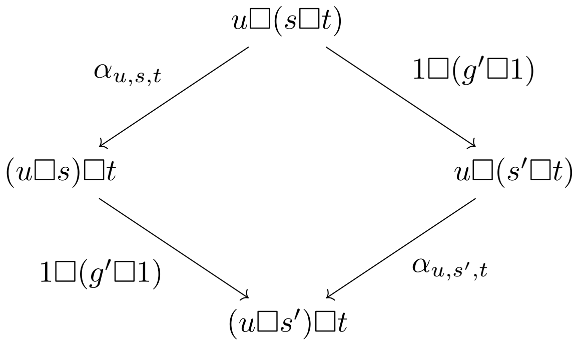

Fig.8 Directed arrows in the Case 5-1

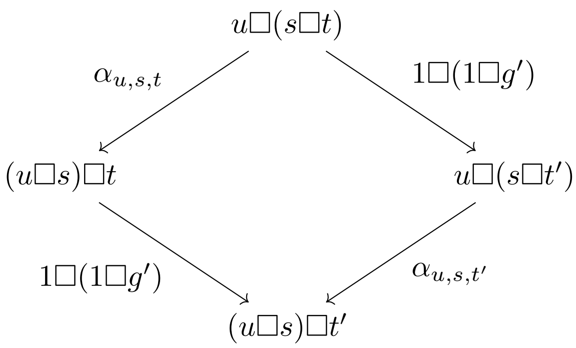

Fig.9 Directed arrows in the Case 5-2

of inverse directed arrows from $w^{(n)}$ to w. Since Fig. 11 is commutative, every arrow from v to w is a composition of a composition of directed arrows from v to $w^{(n)}$ and inverse directed arrows

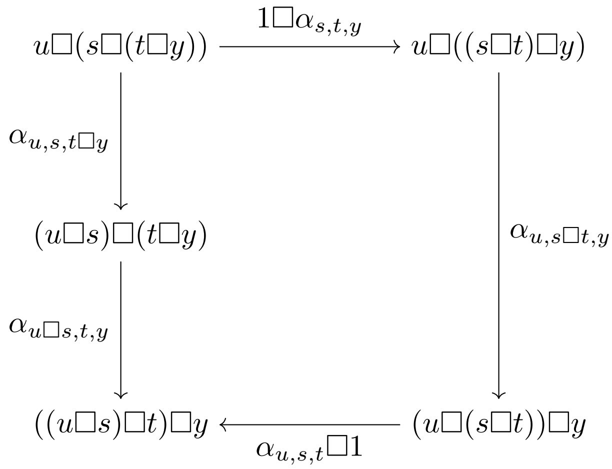

Fig.10 Directed arrows in the Case 5-3

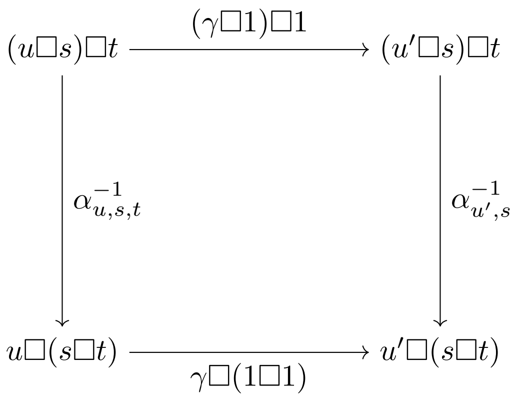

Fig.11 A path where a directed arrow occurs before an inverse directed arrow - case 1

from $^{(n)}$ to w, in that order. Here, we assume that γ is a directed arrow. When $(\gamma\square 1)\square 1$ in Fig. 11 is $(1\square\gamma)\square 1$ and likewise when it is $(1\square 1)\square\gamma$, the diagram is also commutative. And Fig. 12 is commutative by Condition. 8 of Definition 1.

Therefore, every arrow from v to w is equal to the composition of a directed path from v to $w^{(n)}$ and an inverse directed path from $w^{(n)}$ to w, and

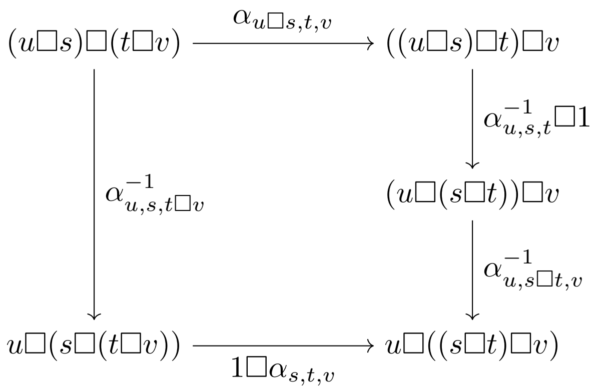

Fig.12　A path where a directed arrow occurs before an inverse directed arrow - case 2

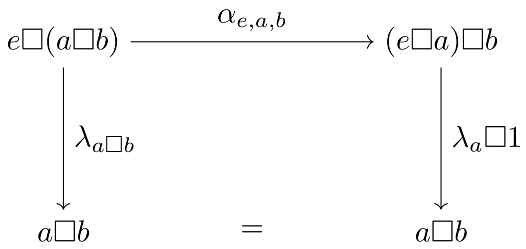

Fig.13　Commutativity 1 of λ, ϱ in monoidal category

this arrow is unique. □

Lemma 3.
Let v be any object in $\mathbf{W}$ that contains e_0, and let v' be the word obtained by removing all occurrences of e_0 from v. Then there exists a unique composition of directed arrows from v to v' except when $v = v'$.

If $v = e_0\Box(((-)\Box(e_0\Box(-))))\Box(-)$, then $v' = ((-)\Box(-))\Box(-)$. To prove this lemma, we first prove the following lemma.

Lemma 4.
Let $\langle \mathbf{C}, \Box, e, \alpha, \lambda, \varrho\rangle$ be a monoidal category.Then, for any $a, b \in \mathbf{C}$, Fig. 13 and Fig. 14 are commutative.

Proof of Lemma 4. All triangles and rectangles in Fig. 15, except for (1), commute by the definition of a monoidal category and the naturality of α. Then, $\alpha_{e,a,b} \circ 1\Box\lambda_{a\Box b} = \alpha_{e,a,b} \circ 1\Box(\lambda_a\Box 1) \circ 1\Box\alpha_{e,a,b}$. Since α is natural isomorphism, it follows that $1\Box\lambda_{a\Box b} = 1\Box(\lambda_a\Box 1)\circ 1\Box\alpha_{e,a,b}$. Therefore, $\lambda_{a\Box b} = \lambda_a\Box 1 \circ \alpha_{e,a,b}$. This means (1) in Fig. 15 and Fig. 13 are commutative.

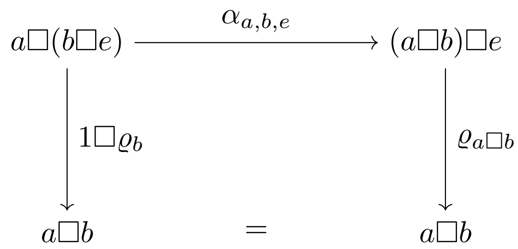

Fig.14　Commutativity 2 of λ, ϱ in monoidal category

Similarly, All triangles and rectangles in Fig. 16, except for (2), commute by the definition of a monoidal category and the naturality of α. Then, $\varrho_{a\Box b}\Box 1 \circ \alpha_{a,b,e}\Box 1 \circ \alpha_{a,b\Box e,e} \circ 1\Box\alpha_{b,e,e} = (1\Box\varrho_b)\Box 1 \circ \alpha_{a,b\Box e,e} \circ 1\Box\alpha_{b,e,e}$. Therefore, $\varrho_{a\Box b} \circ \alpha_{a,b,e} = 1\Box\varrho_b$. This means (2) in Fig. 16 and Fig. 14 are commutative.

□

Proof of Lemma 3. We prove it by induction on the number n of occurrences of components of α in the composition of directed arrows from v to v'. When $n = 0$, since Fig. 17 through Fig. 21 commute, the composition remains equal even if the order of any adjacent λ and ϱ in the directed path from v to v' is swapped. In Fig. 17 through Fig. 21, let β and γ be directed arrows containing either λ or ϱ.

When $n \geq 1$, If an component of α does not contain e_0 in the domain and codomain, then it lowers the rank below that of v', and therefore any component of α must contain e_0 in the domain and codomain. Moreover, since v' does not contain e_0, there exists a path from v to v' in which a directed arrow containing λ or ϱ follows immediately after an component of α. Since Fig. 13, Fig. 14 and Fig. 2 is commutative, one occurrence of α can be eliminated from the directed path from v to v'. Therefore, the composition of directed arrows from v to v' is uniquely determined.

□

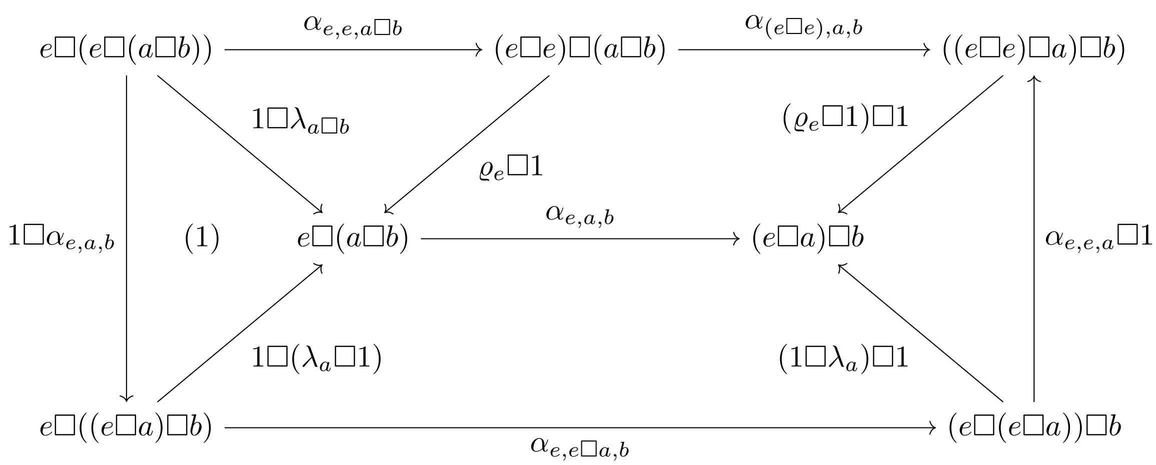

Fig.15　Commutativity 1 for lemma 4

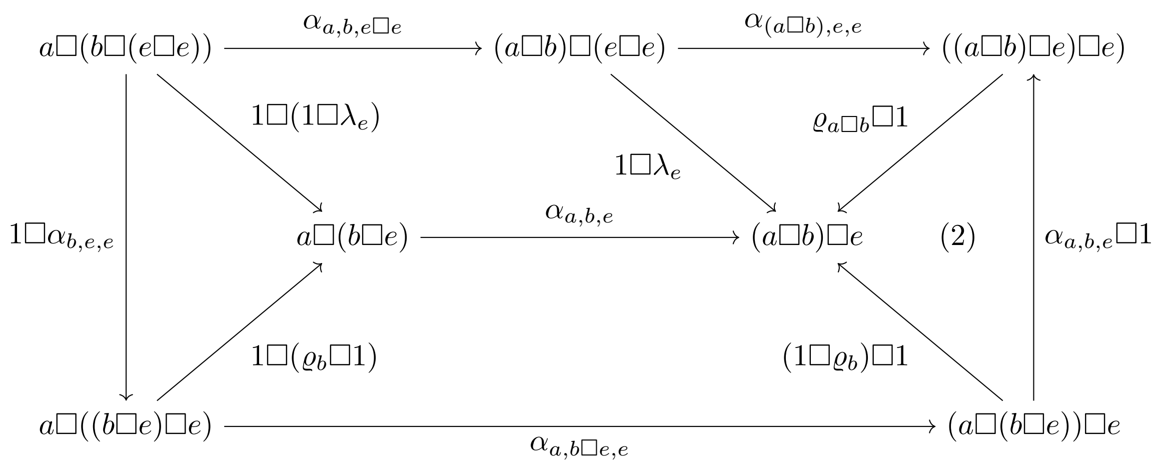

Fig.16　Commutativity 2 for lemma 4

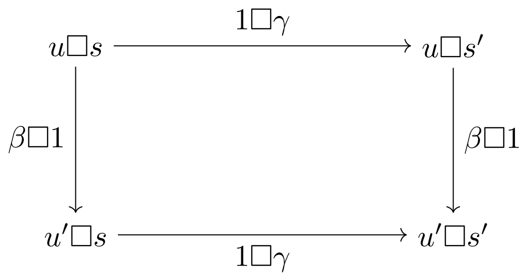

Fig.17　Commutativity 1 of λ and ρ

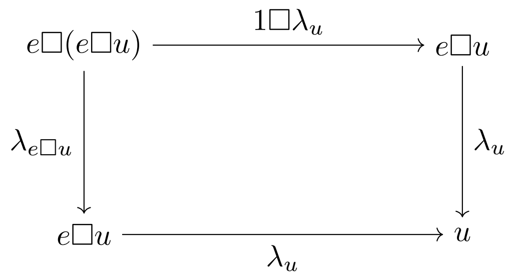

Fig.19　Commutativity 3 of λ and ρ

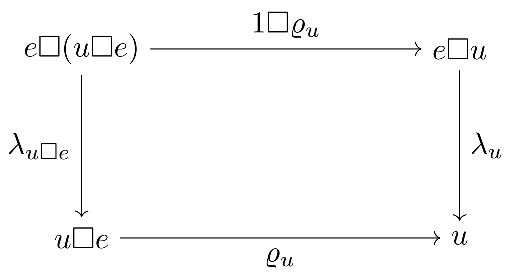

Fig.18　Commutativity 2 of λ and ρ

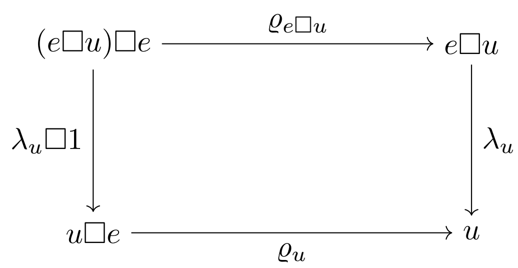

Fig.20　Commutativity 4 of λ and ρ

Proof of Theorem 1. Let β be directed arrows containing either component of λ or ϱ. Fig. 22 through Fig. 26 are all commutative. Then, every arrow from v to w is equal to the composition of a directed path from v to v', a directed path from v' to $w^{(n)}$, an inverse directed path from $w^{(n)}$ to w' and an in-

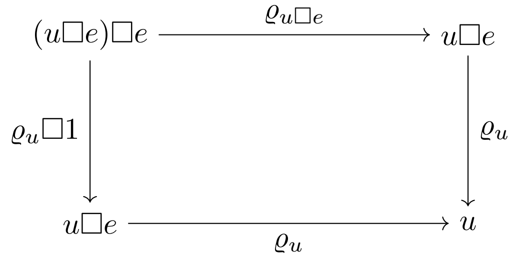

Fig.21　Commutativity 5 of λ and ρ

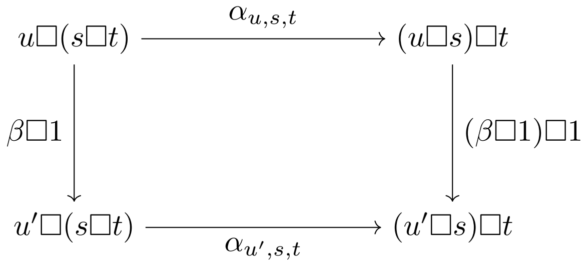

Fig.22　Commutativity 1 of α and directed arrrows

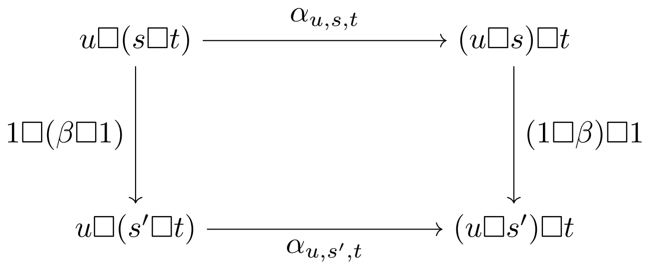

Fig.23　Commutativity 2 of α and directed arrrows

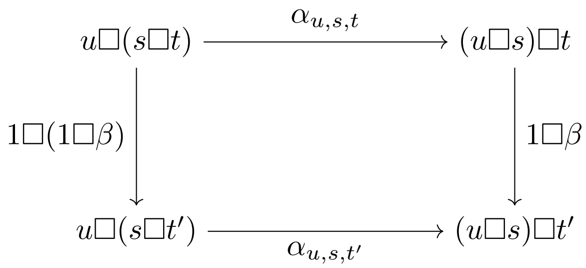

Fig.24　Commutativity 3 of α and directed arrrows

verse directed path from w' to w. Here we assume that w' is the word obtained by removing all occurrences of e_0 from w. Therefore, combining the results of Lemma. 1 and Lemma. 3 , Theorem. 1 is proved. □

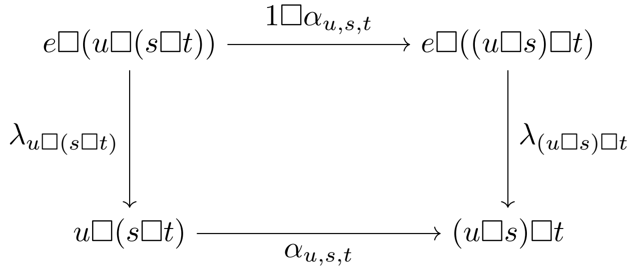

Fig.25　Commutativity 4 of α and directed arrrows

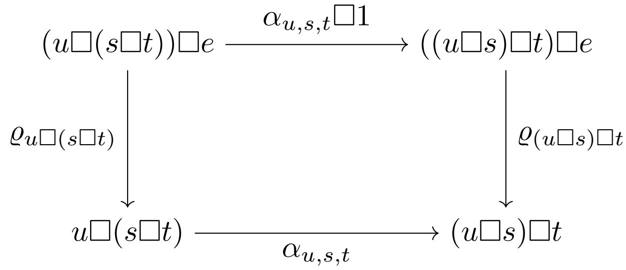

Fig.26　Commutativity 5 of α and directed arrrows

Theorem 2 (Coherence Theorem)**.**
Let $\langle \mathbf{C}, \Box', e', \alpha', \lambda', \varrho' \rangle$ *be arbitary monoidal category, and c be an arbitary object of* $\mathbf{C}$*. Then, there exists a unique morphism of monoidal categories from* $\langle \mathbf{W}, \Box, e_0, \alpha, \lambda, \varrho \rangle$ *to t* $\langle \mathbf{C}, \Box', e', \alpha', \lambda', \varrho' \rangle$ *that maps* $(-) \in \mathbf{W}$ *to c.*

Proof of Theorem 2. Let $()_c$ be a functor from $\mathbf{C}$ to $\mathbf{W}$. For $()_c$ to be a morphism in monoidal categories, it is necessary that $(e_0)_c = e'$. Moreover, since $(-)_c = c$, for any $v, w \in ob(\mathbf{W})$ is uniquely determined as follows. $(v \Box w)_c = v_c \Box' w_c$. Therefore, for any functor satisfying the conditions in the theorem, the assignment of objects is uniquely determined.

As shown in Theorem 1, all allows in $\mathbf{W}$ are basic arrows and compositions of basic arrows. Therefore, by Conditions 2, 4 and 5 of Definition. 2 of morphism of monoidal categories, the assignment of arrows is uniquely determined. □

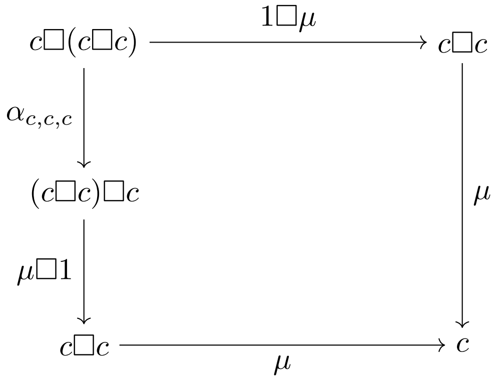

Fig.27 Commutativity of μ of monoid

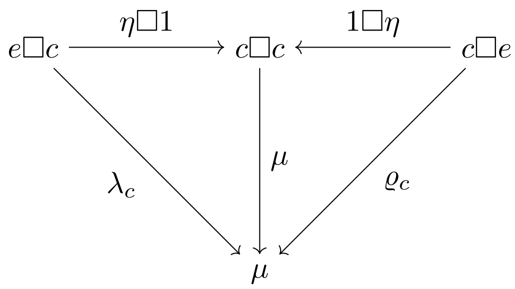

Fig.28 Commutativity of μ and η of monoid

4. General Associative Law in Monoid

Definition 7.

Let $\langle \mathbf{C}, \Box, e, \alpha, \lambda, \varrho\rangle$ be a monoidal category. A monoid $\langle c, \mu, \eta\rangle$ in $\langle \mathbf{C}, \Box, e, \alpha, \lambda, \varrho\rangle$ is defined as follows.

1. *c is an object of $\mathbf{C}$.*
2. *μ is an arrow from $c\Box c$ to c in $\mathbf{C}$ that makes Fig. 27 commute.*
3. *η is an arrow from e to c in $\mathbf{C}$ that makes Fig. 28 commute.*

Definition 8.

Let $\langle \mathbf{C}, \Box', e', \alpha', \lambda', \varrho'\rangle$ be a monoidal category and $\langle c, \mu, \eta\rangle$ be a monoid in $\mathbf{C}$. For each object $w \in ob(\mathbf{W})$, w-fold product $w_c \xrightarrow{\mu_w} c$ in $\mathbf{C}$ is defined as follows.

1. *$\mu_{e_0} = \eta : e \to c$.*
2. *$\mu_{(-)} = 1_c : c \to c$.*
3. *$\mu_{(-)\Box(-)} = \mu : c\Box c \to c$.*
4. *For any $v, w \in of(\mathbf{W})$, $\mu_{v\Box w} = \mu\circ(\mu_v\Box\mu_w) : v_c\Box w_c \to c$.*

Theorem 3 (General Associative Law)**.**

Let $\langle \mathbf{C}, \Box', e', \alpha', \lambda', \varrho'\rangle$ be a monoidal category and $\langle c, \mu, \eta\rangle$ be a monoid in $\mathbf{C}$. For any $v, w \in ob(\mathbf{W})$ such that $length(v) = length(w)$, we have $\mu_w \circ (CAN(v, w))_c = \mu_v$. Here, $(CAN(v, w))c$ denotes the $(f)_c$, where f is the unique arrow from v to w in W. $(-)_c$ is moriphism of monoidal categories defined in Theorem. 2.

Proof of Theorem 3. We prove this by induction on l, the number of basic arrows that constitute $CAN(v, w)$. First, consider the case $l = 1$. If $CAN(v, w)$ is a component of α, let v be $v = u\Box(s\Box t)$. $\mu_{u\Box(s\Box t)} = \mu\circ(\mu_u\Box\mu\circ(\mu_s\Box\mu_t)) = \mu\circ(1\Box\mu)\circ(\mu_u\Box(\mu_s\Box\mu_t))$. And $\mu_{(u\Box s)\Box t} = \mu\circ(\mu\circ(\mu_u\Box\mu_s)\Box\mu_t) = \mu\circ(\mu\Box 1)\circ((\mu_u\Box\mu_s)\Box\mu_t)$. Since Fig. 29 is commutative by naturality of α and Condition. 2 of Definition. 7, $\mu_w \circ CAN(v, w) = \mu_v$ holds. Since α is a natural isomorphism, the statement also holds when $CAN(v, w)$ is a component of α^{-1}.

If $CAN(v, w)$ is a component of λ, let v be $v = e\Box u$. $\mu_{e\Box u} = \mu\circ(\eta\Box\mu_u) = \mu\circ(\eta\Box 1)\circ 1\Box\mu_u = \lambda'_c \circ 1\Box\mu_u$. Since the Fig. 30 is commutative by naturality of λ, $\mu_w \circ CAN(v, w) = \mu_v$ holds. The statement also holds when $CAN(v, w)$ is a component of λ^{-1} by naturality of λ.

If $CAN(v, w)$ is a component of ϱ, let v be $v = u\Box e$. $\mu_{u\Box e} = \mu\circ(\mu_u\Box\eta) = \mu\circ(1\Box\eta)\circ\mu_u\Box 1 = \varrho'_c\circ\mu_u\Box 1$. Since the Fig. 31 is commutative by naturality of ϱ, and $\mu_w\circ CAN(v, w) = \mu_v$ holds. The statement also holds when $CAN(v, w)$ is a component of ϱ^{-1} by naturality of ϱ. Therefore, the claim holds in the case $l = 1$.

Assume that the claim holds for all $l = n - 1$ with $n \geq 2$. For the case $l = n$,let $v = v_1$ and $w = v_{n+1}$, $(CAN(v_1, v_{n+1}))_c = \beta_n\circ\beta_{n-1}\circ\cdots\circ\beta_1$, $v_1 \xrightarrow{\beta_1} v_2 \to \cdots \xrightarrow{\beta_n} v_{n+1}$. By the inductive hypothesis, $\mu_{v_{n+1}}\circ\beta_n\circ\cdots\circ\beta_2 = \mu_{v_2}$ and $\mu_{v_2}\circ\beta_1 = \mu_{v_1}$. Therefore, $\mu_{v_{n+1}}\circ\beta_n\circ\cdots\circ\beta_1 = \mu_{v_1}$.

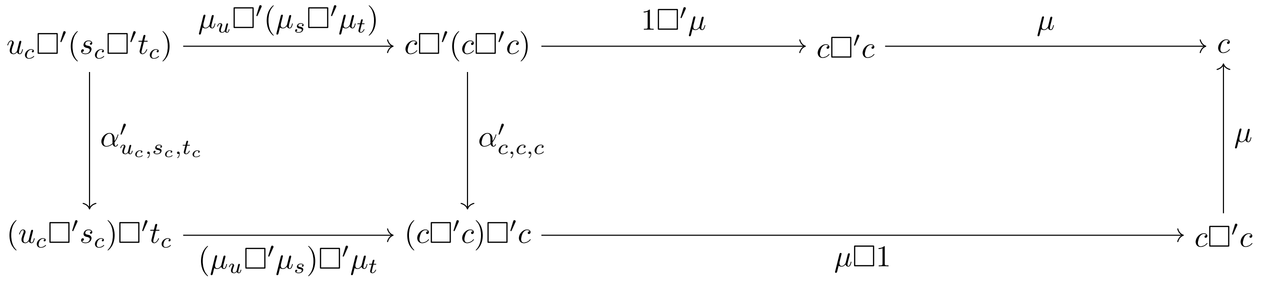

Fig.29　Commutativity when $CAN(v, w) = \alpha$

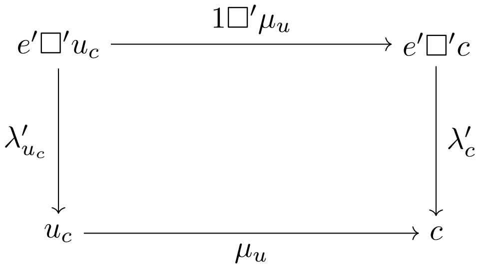

Fig.30　Commutativity when $CAN(v, w) = \lambda$

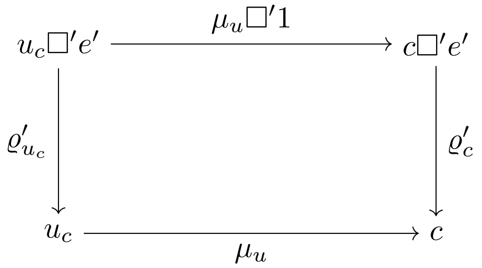

Fig.31　Commutativity when $CAN(v, w) = \varrho$

This completes the proof of the theorem.

□

5. Conclusion

In this paper, we provide a concise and short proof of the Coherence Theorem. In particular, we define all arrows in a category **W** of binary words to be either basic arrows or compositions of basic arrows, and prove that for any $v, w \in ob(\mathbf{W})$ of the same length, there exists a unique arrow in **W** from v to w. Using this theorem and the definition of morphisms in a monoidal category, we derive a proof of the Coherence Theorem. Finally, as an application of the Coherence Theorem, we demonstrate the general associative law for monoids. The Coherence Theorem also implies theorem of construction of free monoid in monoidal category, and so on.

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